Ambedkar University Delhi

Exam I: MPhil and Phd Programme in Mathematics

Time: 3 hrs **25 July 2017** **Max marks: 100**

Do not forget to write your **name** on the booklet. Both Part A and Part B are compulsory.

PART A

Tick the single correct answer in the following questions. Use only a pen to tick answers. Overwriting in the answer to an objective question will mean zero marks for that question. Each part carries $1\frac{1}{2}$ marks.

1. Let G be a group and let a, b be elements of order m and n respectively. Then
   (a) ab has order mn;
   (b) ab has order $m+n$;
   (c) ab has order mn if $ab = ba$;
   (d) ab has order mn if $ab = ba$ and m and n are coprime.

2. Let $G = S_n$ denote the permutation group on n letters. Then
   (a) G has an element of order pq if p and q are distinct primes such that $p + q < n$;
   (b) G has an element of order pq which is even, if p and q are distinct primes such that $p + q < n$ but $2p + q > n$;
   (c) G has an element of order p^2 if $2p < n$;
   (d) G has an element of order p^2 which is even if $2p < n$.

3. Let $V(\mathbb{R})$ be a space of all $n \times n$ skew symmetric matrices with real entries which are such that the sum of the entries in the first row is zero. Then
   (a) $dim(V) = \frac{n(n-1)}{2} + 1$;
   (b) $dim(V) = \frac{n(n-1)}{2} - 1$;
   (c) $dim(V) = \frac{n(n+1)}{2} - 1$;
   (d) $dim(V) = \frac{n(n+1)}{2} + 1$.

4. Let A be a non-zero nilpotent matrix with entries from a field F. Then
   (a) A is similar to a diagonal matrix;
   (b) A is similar to a scalar matrix;
   (c) A cannot be similar to a diagonal matrix;
   (d) A is similar to an upper triangular matrix with at least one non-zero entry in its diagonal.

5. The function $f(z) = |z|^2 + i\bar{z} + 3$ is differentiable at
   (a) i;

(b) -1;

(c) $-i$;

(d) no point in $\mathbb{C}$.

6. Let $f(z)$ be an analytic function with zeros at $z = 1, 2$ and poles at $z = 0, -1$, respectively. Residue of $\frac{f'(z)}{f(z)}$ at $z = 0$ is given by

   (a) 1;

   (b) -1;

   (c) 0;

   (d) 2.

7. Find the remainder obtained on dividing the sum $1! + 2! + 3! + ... + 99! + 100!$ by 12.

   (a) 7;

   (b) 8;

   (c) 9;

   (d) 11.

8. Which of the following statements is correct?

   (a) The Lebesgue measure of every infinite set is infinite;

   (b) The Lebesgue measure of any uncountable set is positive;

   (c) The Lebesgue measure of every bounded measurable set is finite;

   (d) The interval $(0, \infty)$ is a non-measurable set.

9. Identify the correct statement.

   (a) A finite dimensional subspace of a normed linear space need not be complete;

   (b) Every finite dimensional subspace of a normed linear space is closed;

   (c) Every finite dimensional normed linear space is compact;

   (d) None of the above.

10. Consider the ring $R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}_6 \right\}$. Then

    (a) R has only left ideals;

    (b) R has only right ideals;

    (c) R has exactly two proper ideals;

    (d) R has no proper ideals.

11. Consider the polynomials $p(x) = 8x^3 - 6x^2 + 3x - 6$ and $q(x) = 8x^2 + 2x - 3$ in $\mathbb{Z}[x]$. Then

    (a) $p(x)$ and $q(x)$ are both irreducible over $\mathbb{Q}$;

    (b) $p(x)$ and $q(x)$ are both reducible over $\mathbb{Q}$;

    (c) $p(x)$ is irreducible over $\mathbb{Q}$ but $q(x)$ is reducible over $\mathbb{Q}$;

(d) $p(x)$ is reducible over $\mathbb{Q}$ but $q(x)$ is irreducible over $\mathbb{Q}$.

12. Let R be a ring and M be an R-module. Then

 (a) Torsion(M) is a submodule of M;
 (b) Torsion(M) is a submodule of M if R is an integral domain;
 (c) Annihilator of Torsion(M) is a proper ideal of R;
 (d) Annihilator of Torsion(M) is only a left ideal of R.

13. Let $(X, \mathcal{T})$ be a Topological space.

 (a) Every closed, bounded subset of X is compact;
 (b) X is path connected if X is connected;
 (c) If $X = \{a, b, c\}$ and $\mathcal{T} = \{\emptyset, X, \{b\}, \{c\}, \{b, c\}\}$, then X is connected;
 (d) If X is the real line with the usual topology then X is compact.

14. What is the differential equation corresponding to the family of curves $y = k(x-k)^2$, where k is an arbitrary constant

 (a)

$$\left(\frac{dy}{dx}\right)^3 + 4\,x\,y^2\left(\frac{dy}{dx}\right) - 8\,y^2 = 0;$$

 (b)

$$\left(\frac{dy}{dx}\right)^3 - 4\,x\,y\left(\frac{dy}{dx}\right) + 8\,y^2 = 0;$$

 (c)

$$\left(\frac{dy}{dx}\right)^3 - x^2\left(\frac{dy}{dx}\right)^2 + 2y\left(\frac{dy}{dx}\right) + 4y^2 = 0;$$

 (d) None of the above.

15. Consider the following statements with respect to the differential equation

$$\left(2xy\frac{dy}{dx}\right) = y^2 - x^2;$$

 1 The differential equation is a homogeneous differential equation.
 2 The curve represented by the differential equation is a family of circles.
 3 The equation of its orthogonal trajectory is

$$\left(\frac{dy}{dx}\right) = \frac{2xy}{x^2 - y^2}.$$

Which of the following is correct?

 (a) 1 and 2 only;

(b) 1 and 3 only;

(c) 2 and 3 only;

(d) 1, 2 and 3.

16. The series $\frac{\sin x}{1^3} - \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3} - \cdots$ is

(a) divergent;

(b) convergent;

(c) oscillatory;

(d) none of these.

17. Rate of convergence of method of false position is

(a) 1;

(b) 2.32;

(c) 1.62;

(d) 2.

18. Let X be a complete metric space. Then

(a) X is closed and bounded;

(b) Every closed and bounded subset of X is compact;

(c) Every compact subset of X is complete;

(d) Every subset of X that is complete is compact.

19. Let $F \subseteq K \subseteq L$ be fields such that L is an algebraic extension of F.

(a) If L is a normal extension of K and K is a normal extension of F then L is a normal extension of F;

(b) If L is a normal extension of F then K is a normal extension of F;

(c) If L is a normal extension of F then L is a normal extension of K;

(d) If L is a normal extension of K then L is a normal extension of F.

20. The partial differential equation

$$x^2(y-1)u_{xx} - x(y^2-1)u_{xy} + y(y^2-1)u_{yy} + u_x = 0$$

is hyperbolic in the region

(a) $x \neq 0, y > \frac{1}{3}$ *and* $y \neq 1$;

(b) $x \neq 0, y > \frac{1}{3}$ *and* $y \neq -1$;

(c) $x \neq 0, y > \frac{1}{3}$ *and* $y \neq \pm 1$;

(d) none of these.

PART B

Questions 1 and 2 are compulsory. Choose any TWO from 3, 4, 5 and 6. Write answers in the pages after the questions. Justify your answers. Each question carries $17\frac{1}{2}$ marks.

1. Let G be a group that has only finitely many subgroups. Show that G has to be a finite group. If G has exactly 3 subgroups then show that G must be cyclic of order p^2 where p is a prime. Give examples of two cyclic groups G of smallest orders which have exactly 4 subgroups. Also give examples of non-abelian groups of those same orders. Do they have more than 4 subgroups?

2. Using the Method of Residues, show that

$$\int_0^\infty \frac{x^{a-1}}{1+x} dx = \frac{\pi}{2\pi \sin(a\pi)}.$$

   What restrictions must be placed on a?

3. Let $V(\mathbb{R}) = \{P(x) \mid deg(P(x)) \leq 4\}$ and T be a linear operator on V such that

$$T(P(x)) = \int_0^2 P(x)dx$$

   for all $P \in V$.

   (a) Find a basis of the null space of T.

   (b) Find a basis for the range of T.

   (c) Verify the Rank-Nullity Theorem.

4. Consider the ring $R = \mathbb{R}[x, y]$. Show that

   (a) the ideal $< x >$ is prime ideal.

   (b) the ideal $< x, y >$ is maximal ideal.

5. If p is an odd prime, then prove that $(-2/p) = \begin{cases} 1, & \text{if } p \equiv 1 \pmod 8 \text{ or } p \equiv 3 \pmod 8; \\ -1, & \text{if } p \equiv 5 \pmod 8 \text{ or } p \equiv 7 \pmod 8. \end{cases}$

   where (a/p) denotes the Legendre symbol.

6. Solve the initial value problem

$$\left(\frac{d^2y}{dt^2}\right) + y = e^{-2t}\sin t;$$

$$y(0) = 0$$

$$y'(0) = 0$$

   using the Laplace Transform method.