Ambedkar University Delhi

Exam II: Phd Programme in Mathematics

Time: 1hr 30 min **25 July 2017** **Max marks: 100**

Do not forget to write your **name** on the booklet. Answers are to be written in the pages after the questions.

Attempt any Two.

1. If $T(z) = \frac{az+b}{cz+d}$, find necessary and sufficient conditions that $T(\Gamma) = \Gamma$ where Γ is the unit circle $\{z \mid |z| = 1\}$.

2. Let G be a finite group of order $n = p^k m$ where p is a prime, $k \geq 1$ and p does not divide m. Let $S = \{P \subseteq G \mid |P| = p^k\}$.

   (a) Show that G acts on S by left multiplication where $g \cdot P \mapsto gP$. Further show that each orbit O_i will contain an element P_i of S such that $e \in P_i$.

   (b) Show that if G_i is the stabiliser of P_i then $P_i \leq G$ if and only if $G_i = P_i$.

   (c) Show that $|S| \equiv m \pmod{p}$ by considering G to be a cyclic group of order n.

   (d) If n_k is the number of subgroups of order p^k in G, where G is any arbitrary group of order n, then show that $n_k \equiv 1 \pmod{p}$. Hence conclude that G has a Sylow p-subgroup.

3. Let A be the direct product ring $\mathbb{C} \times \mathbb{C}$. Let τ_1 denote the identity map on $\mathbb{C}$ and let τ_2 denote complex conjugation. For any pair $p, q \in \{1, 2\}$ define $f_{p,q} : \mathbb{C} \to \mathbb{C} \times \mathbb{C}$ by $f_{p,q}(z) = (\tau_p(z), \tau_q(z))$.

   (a) Prove that each $f_{p,q}$ is an injective ring homomorphism and that they all agree on the subfield $\mathbb{R}$ of $\mathbb{C}$. Deduce that A has four distinct $\mathbb{C}$-module structures.

   (b) Prove that if $f_{p,q} \neq f_{p',q'}$ then the identity map on A is not a $\mathbb{C}$-module homomorphism, from A (considered as a $\mathbb{C}$-module via $f_{p,q}$), to A (considered as a $\mathbb{C}$-module via $f_{p',q'}$).

4. Let M be a metric space and let X and Y be non-empty subsets of M. Define

$$d(X, Y) = \inf\{d(x, y) \mid x \in X, y \in Y\}\,.$$

   (a) If $X = \{x\}$, Y is closed then prove that there exists $y \in Y$ such that $d(X, Y) = d(x, y)$.

   (b) If X is compact and Y is closed then prove that there exists $x \in X$ and $y \in Y$ such that $d(X, Y) = d(x, y)$.

   (c) Show by example that the conclusion of Part (b) can be false if X and Y are closed but not compact.